Joe Dinius MATH 527B
HW 8 May 16, 2009

1. Problems 1.3.14 and 1.3.11, pg.II-307

   (1.3.14) Let f be a continuous function from a compact metric space (M, d) to a metric space (M', d'). Show that the range $f(M)$ is a compact subset of M'.

   Solution. Let U_α be an open set in M', for some $\alpha \in A$, some indexing set, s.t. $\bigcup_{\alpha \in A} U_\alpha$ is an open cover of M.

   Since f is continuous from $M \to M'$, $f^{-1}(U_\alpha)$ is open in M and the union $\bigcup_{\alpha \in A} U_\alpha$ is an open covering of M. Since M is compact, $\exists$ a finite subcover

$$\bigcup_{i=1}^{k} f^{-1}(U_{\alpha_i}) = M.$$

   Therefore,

$$\begin{aligned} f(M) &= f\left(\bigcup_{i=1}^{k} f^{-1}(U_{\alpha_i})\right) \\ &= \bigcup_{i=1}^{k} f\left(f^{-1}(U_{\alpha_i})\right) \\ &\subseteq \bigcup_{i=1}^{k} U_{\alpha_i}. \end{aligned}$$

   The last two lines of the above argument follow directly from a HW assignment during the first semester. Therefore, $\bigcup_{i=1}^{k} U_{\alpha_i}$ is a finite subcover of $f(M)$. Since such a subcover exists, $f(M)$ is a compact subset of M'.

   (1.3.11) Recall from Example 1.3.3E that the unit ball B in $l^2(\mathbb{R}, \mathbb{N})$ is not compact. Find a continuous function $f : B \to \mathbb{R}$ whose range has least upper bound 1, but which satisfies $f(\mathbf{x}) < 1$ for all $\mathbf{x} \in B$. (Hint: try something like $f(\mathbf{x}) = \sum_{j=0}^{\infty} a_j^2 x^2$, with suitable a_j)

   Solution. Consider $\mathbf{x} \in B_1(0)$. So, for all such $\mathbf{x}$,

$$\sum_{i=1}^{\infty} |x_j|^2 \leq 1.$$

   Utilizing the hint given, pick $a_j = 1 - \frac{1}{j}$. Then

$$\begin{aligned} f(\mathbf{x}) = \sum_{i=1}^{\infty} a_j^2 x_j^2 &\leq ||a_j^2||_\infty ||x_j^2||_1 \\ &\leq \sup_j \left(1 - \frac{1}{j}\right)^2 (1) \\ &< 1 \;\; \forall j \in \mathbb{N}. \end{aligned}$$

Clearly, the sequence $\{a_j\} \to 1$ from below (as $\frac{1}{j} \to 0$). Hence 1 is a least upper bound for range(f), with $f(\mathbf{x}) < 1 \ \forall \mathbf{x} \in B_1(0)$.

2. Problems 1.3.12 and 1.3.13, pg.II-307

   (1.3.12) Prove: if A is a compact subset of a metric space, then there are points $a, b \in A$ whose distance is maximal, i.e. $d(a,b) \geq d(x,y)$ for all $x, y \in A$.

   Proof. Let A be a compact subset of a metric space (M,d). We next show that $d(x,y)$ is bounded above for $x, y \in A$. Suppose not; then for each $n \in N$ there exist $x_n, y_n \in A$ such that $d(x_n, y_n) > n$. By compactness, there are convergent subsequences $\{x_{n_j}\}$ and $\{y_{n_k}\}$ with $x_{n_j} \to a$ and $y_{n_k} \to b$ for some $a, b \in A$. By the triangle inequality, $d(x_{n_j}, y_{n_k}) \leq d(x_{n_j}, a) + d(a,b) + d(y_{n_k}, b) \to d(a,b)$. This is a contradiction, so $d(x,y)$ is bounded above.

   Since the set $\{d(x,y) \mid x, y \in A\}$ is bounded from above, it has a least upper bound L. For each $n > 0$, there are $x_n, y_n \in A$ such that $L - 1/n \leq d(x_n, y_n) \leq L$. By compactness, there are convergent subsequences $\{x_{n_j}\}$ and $\{y_{n_k}\}$ with $x_{n_j} \to a$ and $y_{n_k} \to b$ for some $a, b \in A$. By construction, $d(x_{n_j}, y_{n_k}) \to L$. By the triangle inequality, $d(x_{n_j}, y_{n_k}) \leq d(x_{n_j}, a) + d(a,b) + d(y_{n_k}, b) \to d(a,b)$ and so $d(a,b) = L$. This implies $d(a,b) \geq d(x,y)$ for all $x, y \in A$. □

   (1.3.13) Prove: if A and B are disjoint closed subsets of a metric space, and A is also compact, then the distance between A and B is strictly positive, i.e. there exists a $\delta > 0$ such that $d(a,b) \geq \delta$ for all $a \in A$ and $b \in B$. Show by example that this need not be the case if neither A nor B is compact.

   Solution.

   Proof. We will show this by contradiction. So assume that A and B are disjoint closed subsets of a metric space, and that A is also compact. Further assume that for all $n \in \mathbb{N}$, there exists $a_n \in A$ and $b_n \in B$ such that $d(a_n, b_n) < \frac{1}{n}$. Then $\{a_n\} \subseteq A$ implies that there exists a subsequence $\{a_{n_k}\} \to a$ for some $a \in A$. So fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $d(a_{n_k}, a) < \epsilon/2$ for all $k \geq N$. Thus, for all $k \geq N$,

$$\begin{aligned} d(a, b_{n_k}) &\leq d(a, a_{n_k}) + d(a_{n_k}, b_{n_k}) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

   So $b_{n_k} \to a \in A$. But B closed implies that $a \in B$. Thus, $a \in A \cap B$, so that $A \cap B \neq \emptyset$. This contradicts the assumption that A and B are disjoint.

   As for the second part, let $A = \{a_n\} = \{n\} = \{1, 2, 3 \ldots\}$ and $B = \{b_n\} = \{n - \frac{1}{n}\}$ be subsets of $\mathbb{R}$ with the absolute value topology. Then neither A nor B has an upper bound, so that neither A nor B is compact. Also, we can see that since A and B are disjoint, and that since A and B consist of isolated points, A and B are closed. Fix $\delta > 0$. Then by the Archimedean Property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. So

$d(a_n, b_n) = |a_n - b_n| = |n - (n - \frac{1}{n})| = \frac{1}{n} < \delta$. Thus, the above need not be true if neither A nor B is compact.

Now, an example to show that this need not be the case if neither A nor B is compact. Consider the co-countable topology where the structure of open sets is such that their complements are countable. Therefore, closed sets in this topology are countable. Consider $A, B \subseteq \mathbb{Q}$ s.t. A is the set of all rationals bounded above by $\sqrt{2}$ and B the set of all rationals bounded below by $\sqrt{2}$. Clearly, A^c, B^c are uncountable, therefore A, B are closed wrt the co-countable, but they are not compact, since both sets are unbounded.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $x \in A$ s.t. for all $\epsilon > 0$ $d(x, \sqrt{2}) < \epsilon/2$, and $y \in B$ s.t. $d(y, \sqrt{2}) < \epsilon/2$. Therefore

$$\begin{aligned} d(x,y) &\leq d(x,\sqrt{2}) + d(y,\sqrt{2}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, we have that there $\exists x \in A$, and $\exists y \in B$ s.t. $d(x, y) < \epsilon$.

3. Problems 1.3.16, 1.3.17 and 1.3.18, pg. II-307-308

(1.3.16) Let A be a compact subset of a metric space (M, d). Prove that for every $\epsilon > 0$, there exists a finite subset $S_\epsilon = \{a_1, \ldots, a_{N_\epsilon}\}$ of A with the following property: for every $x \in A$ there is an $a_i \in S_\epsilon$ such that $d(x, a_i) < \epsilon$. (Such a set S_ϵ is called a *finite ϵ-net*)

Proof. Assume that there does not exist such an ϵ-net. Then there exists some $x_1, x_2 \in A$ s.t. $d(x_1, x_2) > \epsilon$. Similarly, there exists an $x_3 \in A$ s.t. $d(x_1, x_3) > \epsilon$ and $d(x_2, x_3) > \epsilon$. Hence, there exists a sequence $\{x_k\}$ s. t.

$$d(x_n, x_m) > \epsilon, n \neq m.$$

However, this implies that there are no convergent subsequences of $\{x_k\}$, since every element is at least ϵ apart.. This contradicts that A is compact.

$\therefore$ for every $x \in A$ there is an $a_i \in S_\epsilon$ such that $d(x, a_i) < \epsilon$.

(1.3.17) This continues the preceding exercise. Suppose A is a closed subset of a complete metric space (M, d) that admits a finite ϵ-net for every $\epsilon > 0$. Show that A is compact.

Solution. By the result in 1.3.16, there is a finite $1/n$-net for any $n \in \mathbb{N}$. Thus for any $\epsilon > 0$ we can choose a $1/n$-net, for $n > 1/\epsilon$, such that any point in M can be approximated by a member of the finite set $S_{1/n}$. Then as $n \to \infty$, $S_{1/n}$ remains countable.

(1.3.18) Let (M, d) be a compact metric space. Prove that there exists a countable dense subset of M. (Such a space is called *separable*, cf. Exercise 1.3.5.39)

Proof. We have already shown that every compact subset of M has a finite $\frac{1}{n}$-net, say A_n, where A_n is a finite set. Then

$$\bigcup_{\alpha \in A_n} B_{\frac{1}{n}}(\alpha) = M$$

where $B_{\frac{1}{n}}(\alpha) = \{x \in M : d(x, \alpha) < \frac{1}{n}\}$. Consider $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$. $\mathcal{A}$ is a countable subset of M, since it is a countable union of finite sets. It remains to show that this set is dense. Given $\epsilon > 0$, $\exists n \in \mathbb{N}$ s.t. $n \geq N \implies \frac{1}{n} < \epsilon$. Take $x \in \bigcup_{\alpha \in A_n} B_{\frac{1}{n}}(\alpha)$. Therefore, there exists $a \in A_n$ s.t. $d(x, a) < \frac{1}{n} < \epsilon$, since A_n is a finite $\frac{1}{n}$-net. Hence, $a \in B_\epsilon(x)$. But we also have that $a \in A_n \subseteq \mathcal{A}$.

$\therefore \mathcal{A}$ is dense in M.

4. In class, we sketched an argument to show that every bounded sequence $\{x_n\}$ in $\mathbb{R}$ has a convergent subsequence by showing that there exists a subsequence $\{x_{n_k}\}$ that converges to $l = \sup(E)$ where

$$E = \{y \in \mathbb{R} \big| x_n > y \text{ for infinitely many indices of } n \in \mathbb{N}\}$$

(a) Show, with all the details, that there is a subsequence $\{x_{n'_k}\}$ that converges to $l' = \inf(E')$ where

$$E' = \{y \in \mathbb{R} \big| x_n < y \text{ for infinitely many indices of } n \in \mathbb{N}\}$$

Solution. This exercise follows almost exactly from results from lecture.

Assume $K = [a, b] \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$. Taking E' as defined above, we conclude that $y < a \implies y \notin E'$. By completeness of $\mathbb{R}$, E' has an inf, call it l' with $l' \geq a$. Additionally, $b \in E' \implies l' \leq b$. Therefore $l' \in K$.

Now, since $\inf(E') = l'$, we can say that $l' - 1/2 \notin K$ while $l' + 1/2 \in K$. This implies that, for an arbitrary bounded sequence $\{x_n\}$, infinitely many terms from this sequence are in $(l' - 1/2, l' + 1/2)$.

We can construct a subsequence now: Consider $m_1 = 25$. There exists $n_1 > m_1$ s.t. $x_{n_1} \in (l' - 1/2, l' + 1/2)$.

Now, take $m_2 = n_1 + 1$, $\exists n_2 > m_2 > n_1$ s.t. $x_{n_2} \in \left(l' - \frac{1}{2^2}, l' + \frac{1}{2^2}\right)$. Follow the same steps inductively. Take $m_k = n_{k-1}+1$, $\exists n_k > m_k > \cdots > m_1$ s.t. $x_{n_k} \in \left(l' - \frac{1}{2^k}, l' + \frac{1}{2^k}\right)$. So, we have constructed a subsequence that converges to l'.

$\therefore$ a convergent subsequence exists that converges to l' in K.

Note: This proof assumes K is a closed *interval.* The corollary from class fills in the holes for K an arbitrary closed set.

(b) Using completeness of $\mathbb{R}$ show that, for any bounded sequence $\{x_n\}$ in $\mathbb{R}$, the following limits exist:

$$\limsup\{x_n\} \equiv \lim_{k\to\infty} \sup_{n\geq k} x_n \quad \liminf\{x_n\} \equiv \lim_{k\to\infty} \inf_{n\geq k} x_n$$

Solution. Consider the sequences

$$\begin{aligned} \{s_n\} &= \{\sup_{k\geq n} x_k\} \\ \{t_n\} &= \{\inf_{k\geq n} x_k\}. \end{aligned}$$

Consider the sequences of such values, $\{s_n\}, \{t_n\}$. Notice that $\{s_n\}$ is monotonically decreasing and $\{t_n\}$ is monotonically increasing. Since $\{x_n\}$ is bounded, $\{s_n\}, \{t_n\}$ are also bounded. Now, we need a theorem.

Theorem. *(Monotone Convergence Thm.)*
If $\{x_n\}$ is monotone and bounded, then $\{x_n\}$ has a finite limit.

Proof. See any undergraduate text on analysis. i.e. Wade's An Introduction to Analysis pg. 44. This proof uses completeness of $\mathbb{R}$.

By the Monotone Convergence Thm., the sequences $\{s_n\}, \{t_n\}$ have finite limits.

$\therefore$ $\limsup\{x_n\}$ and $\liminf\{x_n\}$ exist.

(c) Show directly from the above definitions that there exist subsequences $\{x_{n_k}\}$ and $\{x_{n'_k}\}$ such that $\{x_{n_k}\} \to \limsup x_n$ and $\{x_{n'_k}\} \to \liminf x_n$.

Solution. First, some simple definitions

$$\begin{aligned} s &= \limsup x_n \\ t &= \liminf x_n. \end{aligned}$$

(For the supremum) Set $n_0 = 0$ and let s_n be defined as in previous parts. There exists an $n_1 \in \mathbb{N}$ s.t. $s_{n_0+1} - 1 < x_{n_1} \leq s_{n_0+1}$. Similarly, there is an $n_2 \geq n_1 + 1 > n_1$ s.t. $s_{n_1+1} - 1/2 < x_{n_2} \leq s_{n_1+1}$. Continuing on like this, there exists some $k \in \mathbb{N}$ for which $n_k > \cdots > n_2 > n_1$ s.t.

$$s_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq s_{n_{k-1}+1}.$$

Since $s_{n_{k-1}+1} \to s$, we conclude that $x_{n_k} \to s$.

(For the infimum) Set $n_0 = 0$ and let t_n be defined as in previous parts. There exists an $n_1 \in \mathbb{N}$ s.t. $t_{n_0+1} + 1 > x_{n_1} \geq t_{n_0+1}$. Similarly, there is an $n_2 \geq n_1 + 1 > n_1$ s.t. $t_{n_1+1} + 1/2 > x_{n_2} \geq t_{n_1+1}$. Continuing on like this, there exists some $k \in \mathbb{N}$ for which $n_k > \cdots > n_2 > n_1$ s.t.

$$t_{n_{k-1}+1} + \frac{1}{k} > x_{n_k} \geq t_{n_{k-1}+1}.$$

Since $t_{n_{k-1}+1} \to t$, we conclude that $x_{n_k} \to t$.

(d) $\{x_n\}$ is a bounded sequence, and $\limsup x_n = L$. Show that, for all $\epsilon > 0$, there exists an $N < \infty$ such that $x_n < L + \epsilon$ for all $n \geq N$.

Solution. Since $\{x_n\}$ is bounded, we know that $\{s_n\}$ (see part (b) for definition) is a monotonically decreasing sequence. Therefore, for some $\epsilon > 0$, we can find an $N \in \mathbb{N}$ s.t. $s_n < L + \epsilon$ for all $n \geq N$. Plus, we know that $x_n \leq s_n$ $\forall n$. So,

$$x_n < L + \epsilon \quad \forall n \geq N.$$

(e) If $\liminf x_n \geq \limsup x_n$, show that x_n converges. Conversely, if x_n converges, show that $\liminf x_n = \lim x_n = \limsup x_n$.

Solution. First off, we know that (for s_n, t_n defined as above) $s_n \geq t_n$ for all n. The inequality follows directly for $\liminf x_n$ $\limsup x_n$. Therefore, we conclude that $\limsup x_n = \liminf x_n$. Now to show the desired result.

Pick some $\epsilon > 0$. Let $x = \limsup x_n = \liminf x_n$. Choose $N \in \mathbb{N}$ s.t. $\sup_{k \geq N} x_k - x < \frac{\epsilon}{2}$ and $x - \inf_{k \geq N} < \frac{\epsilon}{2}$. Let $n, m \geq N$ and suppose for simplicity $x_n > x_m$. Then

$$|x_n - x_m| = x_n - x_m \leq \sup_{k \geq N} x_k \ (-x + x) - \inf_{k \geq N} x_k < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $\{x_n\}$ is Cauchy and converges in $\mathbb{R}$. But, from part (c) we know that some subsequence of $\{x_n\}$ converges to x. Therefore, $x_n \to x$.

For the other part of the problem, assume x_n converges. This direction is easier. Suppose $x_n \to x$ Then $x_{n_k} \to x$ for all subsequences x_{n_k}. From part (c) we can conclude that $x = \limsup x_n = \liminf x_n$.

(f) A point $z \in \mathbb{R}$ is an *accumulation point* or equivalently a *cluster point* or a *limit point* of a sequence $\{x_n\}$ if there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\} \to z$. If z is an accumulation point for a bounded sequence $\{x_n\}$ show that

$$\liminf x_n \leq z \leq \limsup x_n.$$

Solution. Suppose $x_{n_k} \to z$ For some $N \in \mathbb{N}$ and choose K large enough so that $k \geq K \implies n_k \geq N$. Clearly,

$$\inf_{j \geq N} x_j \leq x_{n_k} \leq \sup_{j \geq N} x_j$$

for all $k \geq K$. Taking the limit as $k \to \infty$

$$\inf_{j \geq N} x_j \leq z \leq \sup_{j \geq N} x_j.$$

Now, take the limit as $N \to \infty$ and apply the definition of lim sup and lim inf to get

$$\liminf x_n \leq z \leq \limsup x_n.$$

(g) If l and l' are as defined at the beginning of this problem, show that $l = \limsup x_n$ and $l' = \liminf x_n$.

Solution. Looking at the definitions of E and E' from the beginning the problem, we can interpret the intersection of sets $E \cap E'$ as the set of all limits of subsequences of bounded sequences in $\mathbb{R}$. Therefore, we can interpret $l'(l)$ as the smallest (largest) limit(s) of subsequences in $\mathbb{R}$. By definition of lim sup and lim inf we can conclude that $l = \limsup x_n$ and $l' = \liminf x_n$.

5. **Optional** A non-negative sequence x_n is subadditive if $x_{n+m} \leq x_n + x_m$ for all $n, m \in \mathbb{N}$. If x_n is a subadditive sequence, show that $\lim_{n\to\infty} \frac{x_n}{n}$ exists.

   Solution. To see that the sequence is bounded below, just note that $a_n \geq 0$ and so $\frac{a_n}{n} \geq 0$ for all n. Let $l = \inf\left\{\frac{a_n}{n} : n \in \mathbb{N}\right\}$. I want to show that $\frac{a_n}{n} \to l$.

   Let $\epsilon > 0$. Let $K \in \mathbb{N}$, such that $\left|\frac{a_K}{K} - l\right| < \epsilon/2$. There is some such K, since if not $l + \epsilon/2$ would also be a lower bound, contradicting the fact that l is the greatest lower bound.

   Let $M \in \mathbb{N}$ be such that $\frac{a_r}{KM} < \epsilon/2$ for all $r = 0, 1, 2, \ldots, K-1$. To find such an M just find $R = \max\left\{\frac{a_r}{K} : r < K\right\}$ and choose M so that $R/M < \epsilon/2$.

   Let $N = KM$. Pick $n \geq N$ and let $r, s \in \mathbb{N}$ s.t. $n = sK + r$ with $r < K$. So $s \geq M$. Then $\frac{a_n}{n} \leq \frac{sa_K}{sK+r} + \frac{a_r}{sK+r} \leq \frac{sa_K}{sK} + \frac{a_r}{KM} \leq \frac{a_K}{K} + \frac{a_r}{KM} < (l + \epsilon/2) + \epsilon/2$. Since $\frac{a_K}{K} < l + \epsilon/2$. This means that $\left|\frac{a_n}{n} - l\right| < \epsilon$ since $\frac{a_n}{n} \geq l$ by the definition of l.

   $\therefore \lim_{n\to\infty} \frac{x_n}{n}$ exists.

6. **Accumulation points:** $\{x_n\}$ is a given sequence in a topological space A (not necessarily a metric space).

   (a) Is this statement true or false - If $y \in \overline{\{x_n\}}$, then y is an accumulation point for the sequence $\{x_n\}$. Prove or give a counterexample.

   Solution. This statement is false. Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$. The point $y = 1 = x_1 \in \overline{\{x_n\}}$, but 1 is not an accumulation point for the sequence.

   (b) Let F_n denote the closed set $\overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}}$. Show that $y \in F = \bigcap_{n=1}^{\infty} F_n$ iff there is a subsequence x_{n_k} converging to y. (Note: We are not necessarily in a metric space, so you should use the abstract topological definition of convergence).

Solution. ($\Longrightarrow$) First show that if $y \in F$, then there exists a subsequence $\{x_{n_k}\} \to y$. If we assume the topology on A is first countable, then we know there exists a nested sequence of open sets $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \ldots$ containing y s.t. for every open neighborhood O of y, there is an index n s.t. $y \in U_n \subseteq O$. We now construct our subsequence in the following way: The intersection $U_1 \cap F_1$ must be non-empty, since y is contained in both sets. Moreover, since y in the closed set $F_1 = \overline{\{x_n\}_{n=1}^{\infty}}$, any open set containing y must contain an element from an element of $\{x_n\}_{n=1}^{\infty}$. Pick such an element, call it x_{n_1}. Make the same argument inductively: pick an element in the element

$$x_{n_k} \in U_k \cap F_k = \overline{\{x_n\}_{n=1}^{\infty}}.$$

By construction, $\{x_{n_k}\}$ converges y. Let O be any open neighborhood of y. There must be some index N s.t. $y \in U_N \subset O$. Through the nesting of the U_i's, for every index $k \geq N$, we have $x_{n_k} \in U_N \subseteq O$, so O contains the end of the subsequence $\{x_{n_k}\}$ as required.

($\Longleftarrow$) Conversely, that there exists a subsequence $\{x_{n_k}\} \to y$. Let $O \ni y$ be an open set. Since $\{x_{n_k}\}$ converges, there must be some K s.t. $k \geq K \implies x_{n_k} \in O$. So, for any $n \in N$, there exists some index $k \geq K$ s.t. $n_k \geq n$, i.e. there is a subsequence element $x_{n_k} \in F_n$. Since our choice of O was arbitrary, every open set containing y has non-empty intersection with F_n for every n. Then $y \in F_n$ for all n, so $y \in F$.

$\therefore y \in F = \bigcap_{n=1}^{\infty} F_n$ iff there is a subsequence x_{n_k} converging to y.

(c) If the topological space A is compact, show that every sequence in A has a convergent subsequence. (Hint: Finite intersection property. See also, Problem 1.3.33, Pg. II-313).

Solution. Let $\{x_n\} \subset A$ be a sequence in A. Let $F_n = \overline{\{x_n, x_{n+1}, \ldots\}}$. Then $\mathcal{F} = \{F_1, F_2, \ldots\}$ has the finite intersection property. From Proposition 1.3.24 of the class notes, A is compact iff $\mathcal{F}$ has the finite intersection property. The following is true,

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$

That is, there exists some $y \in \bigcap_{F \in \mathcal{F}} F$. Since $y \in F$, for all $F \in \mathcal{F}$, we have that for all k, $F_{n_k} \in \mathcal{F}$ s.t. $y \in \lim_{k \to \infty} F_{n_k}$. Therefore $y \in \overline{\{x_{n_k}, x_{n_k+1}, \ldots\}}$. Since $\{x_{n_k}, x_{n_k+1}, \ldots\}$ is an infinite set, we have that y is an accumulation point of A. Given that A is compact, we have that A contains all of its accumulation points, so $y \in A$. Moreover, $\{x_{n_k}\}$ has a limit in A.

7. Problem 1.3.34, Pg. II-313.

   (1.3.34) Let f be a continuous function from a compact topological space $(X, \mathcal{T})$ to a topological space $(Y, \mathcal{S})$. Then the range of f is a compact subset of Y.

Solution. Let $\{O_\alpha : \alpha \in A\}$ be an open cover of $f(X)$. By continuity of f, $\{f^{-1}(O_\alpha) : \alpha \in A\}$ is open in X. Additionally, the union over all $\alpha \in A$ of these sets is an open cover of X. Since X is compact, we can construct a finite subcover of the form

$$\bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i}) = X, \quad \alpha_i \in A, i = 1, \dots, n.$$

So the image of X becomes

$$\begin{aligned} f(X) &= f\left(\bigcup_{i=1}^{n} f^{-1}(O_{\alpha_i})\right) \\ &= \bigcup_{i=1}^{n} f\big(f^{-1}(O_{\alpha_i})\big) \\ &\subseteq \bigcup_{i=1}^{n} O_{\alpha_i}. \end{aligned}$$

Therefore a finite subcover of $f(X)$ exists, so $f(X)$ is a compact subset of Y.

This problem seemed awfully familiar.

8. Problem 1.3.36, Pg. II-313.

(1.3.36)

Definition. A topological space $(X, \mathcal{T})$ is said to be *locally compact* if every point has a neighborhood whose closure is compact.

Prove that $\mathbb{R}$ is locally compact. Prove that $l^2(\mathbb{R}, \mathbb{N})$ is not locally compact.

Proof. Let $x \in \mathbb{R}$; the interval $(x-1, x+1)$ is an open neighborhood of x and its closure $[x-1, x+1]$ is closed and bounded (compact) in $\mathbb{R}$. Therefore $\mathbb{R}$ is locally compact.

To prove that $l^2(\mathbb{R}, \mathbb{N})$ is not locally compact, we must find a point $x \in l^2(\mathbb{R}, \mathbb{N})$ s.t. every open neighborhood of x has closure which is not compact. Let $x = 0$, and let O be an open set containing 0. Then $\exists$ some open ball B_ϵ (as defined in the usual metric topology sense) s.t. $0 \in B_\epsilon \subset O$ with closure $\bar{B}_\epsilon$. Consider the sequence $x_n = \{0, \dots, \epsilon, \dots\}$, where ϵ is in the nth spot. The norm of this vector is clearly ϵ, however the norm $||x_n - x_m||_2 = \sqrt{2}\epsilon$ for $n \neq m$. This sequence is not Cauchy, so there cannot be a convergent subsequence in $\bar{B}_\epsilon$, which implies $\bar{B}_\epsilon$ is not compact. I claim that $\bar{O}$ cannot be compact, because if it were $\bar{B}_\epsilon \subset \bar{O}$ would have to be compact as well. So, no open neighborhood of 0 has compact closure which implies $l^2(\mathbb{R}, \mathbb{N})$ is not locally compact.